ON GROUND STATES FOR THE L^2-CRITICAL BOSON STAR EQUATION

RUPERT L. FRANK AND ENNO LENZMANN

Abstract. We consider ground state solutions $u \geqslant 0$ for the L^2-critical boson star equation

$$\sqrt{-\Delta}\, u - (|x|^{-1} * |u|^2)u = -u \quad \text{in } \mathbb{R}^3.$$

We prove analyticity and radial symmetry of u.

In a previous version of this paper, we also stated uniqueness and nondegeneracy of ground states for the L^2-critical boson star equation in $\mathbb{R}^3$, but the arguments given there contained a gap. However, we refer to our recent preprint [FraLe] in `arXiv:1009.4042`, where we prove a general uniqueness and nondegeneracy result for ground states of nonlinear equations with fractional Laplacians in $d = 1$ space dimension.

1. Introduction and Main Results

We consider ground states for the *massless boson star equation* in $d = 3$ dimensions given by

$$\begin{cases} \sqrt{-\Delta}\, u - \big(|x|^{-1} * |u|^2\big)u = -u, \\ u \in H^{1/2}(\mathbb{R}^3), \quad u \geqslant 0, \quad u \not\equiv 0. \end{cases} \tag{1.1}$$

Here $H^s(\mathbb{R}^3)$ is the inhomogeneous Sobolev space of order $s \in \mathbb{R}$, and the symbol $*$ denotes the convolution on $\mathbb{R}^3$.

The nonlinear equation (1.1) plays a central role in the mathematical theory of gravitational collapse of boson stars, which we briefly summarize as follows. In the seminal work of Lieb and Yau [LiYa], the universal constant

$$N_* = \|u\|_2^2 \tag{1.2}$$

was found to be the so-called *"Chandrasekhar limiting mass"* for boson stars in the time-independent setting. Here the ground state $u \in H^{1/2}(\mathbb{R}^3)$, appearing in equation (1.2), is a certain optimizer that solves problem (1.1). As one main result, it was proven in [LiYa] that boson stars with total mass strictly less than N_* are gravitationally stable, whereas boson stars whose total mass exceed N_* may undergo a "gravitational collapse" based on variational arguments and many-body quantum theory. Moreover, it was conjectured by Lieb and Yau in [LiYa] as an open problem that uniqueness for ground states holds.

More recently, the mathematical theory of boson stars has entered the field of nonlinear dispersive equations: In [ElSc], it was shown that the *dynamical evolution* of boson stars is effectively described by the nonlinear evolution equation (with mass parameter $m \geqslant 0$)

$$i\partial_t \psi = \sqrt{-\Delta + m^2}\, \psi - \big(|x|^{-1} * |\psi|^2\big)\psi \tag{1.3}$$

for the wave field $\psi : [0,T) \times \mathbb{R}^3 \to \mathbb{C}$. In fact, this dispersive nonlinear L_2-critical PDE displays a rich variety of phenomena such as stable/unstable traveling solitary waves and finite-time blowup. In particular, the ground states $u(x) \geqslant 0$ for (1.1) and the constant $N_* > 0$ given by (1.2) both play a fundamental role as follows: First, the ground state solutions $u(x)$ of (1.1) give rise to *ground state solitary waves* of the form

$$\psi(t,x) = e^{it} u(x) \tag{1.4}$$

for the evolution equation (1.3) in the case of vanishing mass $m = 0$. Second, the universal constant $N_* > 0$ sets the scale between "small" and "large" solutions of the L_2-critical nonlinear dispersive PDE (1.3), irrespectively of the value for $m \geqslant 0$. More precisely, as shown in [FrLe, Le2], all solutions $\psi \in C^t_0 H^{1/2}_x([0,T) \times \mathbb{R}^3)$ with small L_2-mass

$$\|\psi(t)\|_2^2 < N_*$$

extend globally in time (i.e. we have $T = \infty$); whereas solutions with

$$\|\psi(t)\|_2^2 > N_*$$

can lead to blowup at some finite time $T < \infty$. (This singularity formation indicates the dynamical "gravitational collapse" of a boson star.) Thus, any analytical insight into some key properties (e.g., uniqueness up to translation) of the ground states $u(x) \geqslant 0$ and the spectrum of their linearization will be of considerable importance for a detailed blowup analysis for the nonlinear dispersive equation (1.3).

Our main result is as follows.

Theorem 1.1. (Radiality and Analyticity). *Every solution $u \in H^{1/2}(\mathbb{R}^3)$ of problem* (1.1) *is of the form $u(x) = Q(x-a)$ for some $a \in \mathbb{R}^3$, where Q satisfies the following properties.*

(i) *Q is positive, radial and strictly decreasing.*
(ii) *Q is real-analytic. More precisely, there exists a constant $\sigma > 0$ and an analytic function $\tilde{Q}$ on $\{z \in \mathbb{C}^3 : |\operatorname{Im} z_j| < \sigma, 1 \leqslant j \leqslant 3\}$ such that $\tilde{Q}(x) = Q(x)$ if $x \in \mathbb{R}^3$.*

Remark 1. A natural open question is uniqueness of the ground state $Q = Q(|x|) > 0$. We refer to our recent work [FraLe], where uniqueness has been proven for ground state of nonlinear equations with fractional Laplacians $(-\Delta)^s$ in $d = 1$ dimension.

Remark 2. Our proof that any solution of problem (1.1) must be radially symmetric (with respect to some point) employs the classical method of moving planes introduced in [GiNiNi]; see Section 3 below. See also [BiLoWa] for a similar symmetry result for the moving plane method applied to equation with fractional Laplacians on the unit ball $\{x \in \mathbb{R}^3 : |x| < 1\}$. We remark that the arguments, which we present in Section 3 below, are able to deal with the unbounded domain $\mathbb{R}^3$, and thus settling an open problem stated in [BiLoWa].

While finalizing the present paper, we learned that [MaZh] have very recently and independently established a symmetry result for the equation $-\Delta u - \big(|x|^{-1} * |u|^2\big) u = -u$ in $\mathbb{R}^3$. They also briefly sketch [MaZh, Sec. 5] how to extend their approach to more general equations, including (1.1). Their method is different from ours and uses the integral version of the method of moving planes developed in [ChLiOu]. We believe that our non-local Hopf's lemma, on which our differential

version of moving planes is based, might have applications beyond the context of the present paper.

Remark 3. Note that Theorem 1.1 implies an analagous statement for positive solutions of the equation

$$\sqrt{-\Delta}\, u - \kappa(u^2 * |x|^{-1})u = -\lambda u.$$

with constants $\kappa, \lambda > 0$. Indeed, u solves this equation if and only if $\kappa^{-1/2}\lambda^{-3/2}u(x/\lambda)$ solves (1.1). One might also ask whether this equation can have a solution for $-\lambda = E \geqslant 0$. The answer is negative, even if the positivity assumption of u is dropped, as shown by the next result (whose proof is given in Subsection 2.3 below). Without loss of generality, we can put $\kappa = 1$ in the following.

Proposition 1.2. *Let $E \geqslant 0$. If $u \in H^{1/2}(\mathbb{R}^3)$ is radial and satisfies $\sqrt{-\Delta}\, u - (|u|^2 * |x|^{-1})u = Eu$, then $u \equiv 0$.*

Organization of the Paper. In Sections 2–4, we organize the proof of Theorem 1.1 as follows. In Section 2 we collect some preliminary results on (1.1) about the existence, regularity, and spatial decay of solutions. Moreover, we give the proof of Proposition 1.2. In Section 3 we implement the method of moving planes and we prove that every solution of (1.1) is spherically symmetric with respect to some point. In Section 4 we prove the real analyticity of solutions. In Section 5, we provide further analyticity results about elements in the kernel of the linearization of (1.1).

Notation and Conventions. For the Fourier transform on $\mathbb{R}^3$, we use the convention

$$\hat{u}(\xi) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} u(x) e^{-i\xi\cdot x}\, dx. \tag{1.5}$$

As usual, the fractional derivative operators $(-\Delta)^s$ and $(1-\Delta)^s$ are defined via their multipliers $|\xi|^{2s}$ and $(1+|\xi|^2)^s$ in Fourier space, respectively. Lebesgue spaces of functions on $\mathbb{R}^3$ will be denoted by $L_p = L_p(\mathbb{R}^3)$ with norm $\|\cdot\|_p$ and $1 \leqslant p \leqslant \infty$. For the sake of brevity, we shall use the notation $\|u\| \equiv \|u\|_2$ occasionally. We employ inhomogeneous Sobolev norms $\|u\|_{H^s} := \|(1-\Delta)^{s/2}u\|_2$, as well as homogeneous Sobolev norms $\|u\|_{\dot{H}^s} = \|(-\Delta)^{s/2}u\|_2$. The equation (1.1) is always understood to hold in the $H^{-1/2}$ sense. That is, we say that $u \in H^{1/2}(\mathbb{R}^3)$ solves the equation in (1.1) if

$$\int_{\mathbb{R}^3} |\xi| \overline{\hat{v}(\xi)} \hat{u}(\xi)\, d\xi - \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\overline{v(x)} u(x) |u(y)|^2}{|x-y|}\, dx\, dy = -\int_{\mathbb{R}^3} \overline{v(x)} u(x)\, dx\,,$$

for all $v \in H^{1/2}(\mathbb{R}^3)$.

In what follows, the letter C denotes a constant which is allowed to change from inequality to inequality. With the usual abuse of notation, we shall not distinguish between the functions $f(|x|)$ and $f(x)$ whenever $f : \mathbb{R}^3 \to \mathbb{C}$ is radial.

Acknowledgments. R. F. gratefully acknowledges support through DFG grant FR 2664/1-1 and NSF grant PHY 06 52854. E. L. is supported by a Steno fellowship of the Danish research council and NSF grant DMS-0702492.

2. Preliminary results

To prepare the proof of our main results, we first collect some preliminary results on the existence, regularity, and decay of solutions to problem (1.1). Since all these facts follow from the literature and standard arguments, we will keep our exposition brief throughout this section.

2.1. **Existence and properties of a minimizing solution.** The existence of a nonnegative, radial solution $Q(|x|) \geqslant 0$ of problem (1.1) can be established by direct variational arguments, as remarked in [LiYa, App. A.2]. More precisely, we consider the minimization problem

$$\inf \big\{ I[u] : u \in H^{1/2}(\mathbb{R}^3),\, u \not\equiv 0 \big\}, \tag{2.1}$$

where

$$I[u] := \frac{\|(-\Delta)^{1/4} u\|^2 \; \|u\|^2}{\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |u(x)|^2 |x-y|^{-1} |u(y)|^2 \, dx\, dy} \,. \tag{2.2}$$

Thanks to strict rearrangement inequalities (see [Li, FrSe]), we have that $I[u^*] \leqslant I[u]$ with equality if and only if $u(x)$ equals its symmetric-decreasing rearrangement $u^*(|x|) \geqslant 0$ (modulo translation in space and multiplication by a complex number). As pointed out in [LiYa, App. A.2], this fact permits us to imitate the proof in [LiOx] to deduce the existence of a symmetric-decreasing minimizer $Q = Q^* \in H^{1/2}(\mathbb{R}^3)$ for problem (2.1). Moreover, an elementary calculation shows that any minimizer for (2.1) satisfies the Euler-Lagrange equation

$$\sqrt{-\Delta}\, Q - \kappa(|Q|^2 * |x|^{-1}) Q = -\lambda Q,$$

with some constants $\lambda > 0$ and $\kappa > 0$ that both depend on Q. By Remark 3, we see that any symmetric-decreasing minimizer $Q = Q^* \in H^{1/2}(\mathbb{R}^3)$ for (2.1) furnishes (after suitable rescaling) a solution of problem (1.1).

2.2. **Regularity and Decay.** In this subsection, we collect some basic regularity and decay estimates for solutions $u \in H^{1/2}(\mathbb{R}^3)$ of the nonlinear equation

$$\sqrt{-\Delta}\, u - \big(|u|^2 * |x|^{-1}\big) u = -u. \tag{2.3}$$

Note that we do not require u to be non-negative or even real-valued in this section, unless we explicitly say so.

Lemma 2.1 (Smoothness of solutions)**.** *Let $u \in H^{1/2}(\mathbb{R}^3)$ be a solution of* (2.3)*. Then $u \in H^s(\mathbb{R}^3)$ for all $s \geqslant 1/2$.*

Proof. This follows from a simple bootstrap argument. Indeed, note that u satisfies

$$u = (\sqrt{-\Delta} + 1)^{-1} F(u), \tag{2.4}$$

where we put $F(u) = (|u|^2 * |x|^{-1})u$. Since $F(u)$ maps $H^s(\mathbb{R}^3)$ into itself for any $s \geqslant 1/2$ (see, e. g., [Le2]) and thanks to the smoothing property $(\sqrt{-\Delta}+1)^{-1} : H^s(\mathbb{R}^3) \to H^{s+1}(\mathbb{R}^3)$, we obtain the desired result. □

Next, we record a decay estimate for solutions of (2.3).

Lemma 2.2 (Decay rates)**.** *Any solution $u \in H^{1/2}(\mathbb{R}^3)$ of* (2.3) *satisfies*

$$|u(x)| \leqslant C(1+|x|)^{-4} \tag{2.5}$$

and

$$(|u|^2 * |x|^{-1})(x) \leqslant C(1+|x|)^{-1}. \tag{2.6}$$

Moreover, if we assume that $u(x) \geqslant 0$ and $u \not\equiv 0$, then we also have the lower bound

$$u(x) \geqslant C(1+|x|)^{-4}. \tag{2.7}$$

In particular, any such solution $u(x)$ is strictly positive.

Proof. Note that $u \in L_2(\mathbb{R}^3)$ is an eigenfunction for the "relativistic" Schrödinger operator

$$H := \sqrt{-\Delta} + V$$

with the local potential $V(x) = -(|u|^2 * |x|^{-1})(x)$. Furthermore, by Lemma 2.1 and Sobolev embeddings, we have $u \in L_p(\mathbb{R}^3)$ for all $p \geqslant 2$, which implies that $V(x)$ is continuous and $V(x) \to 0$ as $|x| \to \infty$. Hence u is an eigenfunction corresponding to the eigenvalue -1 below the bottom of the essential spectrum of H. From [CaMaSi, Proposition IV.1] we now deduce the bound (2.5).

Next, we see that deriving the bound (2.6) amounts to estimating the function $V(x)$ defined above. First, we note that the Hardy-Kato inequality (see, e. g., [He]) gives us

$$|V(x)| \leqslant \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\, dy \leqslant C \int_{\mathbb{R}^3} \overline{u}(y)(\sqrt{-\Delta}\, u)(y)\, dy \leqslant C\|u\|^2_{H^{1/2}}.$$

Also, from (2.5) we have a radially symmetric bound for $u(x)$. Thus, by Newton's theorem (see, e. g., [LiLo, Theorem 9.7]), we deduce

$$|V(x)| \leqslant \frac{1}{|x|} \int_{\mathbb{R}^3} \frac{C}{(1+|y|)^4}\, dy \leqslant \frac{C}{|x|}.$$

Combining these two estimates for $V(x)$, we obtain the bound (2.6).

Finally, let us also assume that $u(x) \geqslant 0$ and $u \not\equiv 0$. By standard Perron-Frobenius arguments, we conclude that $u(x)$ is the unique ground state eigenfunction for the Schrödinger operator H. In particular, invoking [CaMaSi, Proposition IV.3] yields the lower bound (2.7). □

Remark 4. As an alternative to probabilistic arguments used in [CaMaSi], we could also provide a more "hands-on" proof of Lemma 2.2, which is based on bootstrapping equation (2.4) and using the explicit formula for the Green's function

$$\begin{aligned}(\sqrt{-\Delta}+\tau)^{-1}(x,y) &= \int_0^\infty e^{-t\tau} \exp(-t\sqrt{-\Delta})(x,y)\, dt \\ &= \frac{1}{\pi^2} \int_0^\infty e^{-t\tau} \frac{t}{(t^2+|x-y|^2)^2}\, dt. \end{aligned} \tag{2.8}$$

We refer to [Le3] for details for this alternate proof; see, e. g., [LiLo, Section 7.11] for the explicit formula of the kernel.

2.3. **Proof of Proposition 1.2.** Suppose $u \in H^{1/2}(\mathbb{R}^3)$ is radial and solves

$$\sqrt{-\Delta}\, u - \big(|x|^{-1} * |u|^2\big)u = Eu$$

with some constant $E \geqslant 0$. With $V := -|u|^2 * |x|^{-1}$ one has the virial identity

$$\|(-\Delta)^{1/4}u\|^2 = \int |x| \partial_r V |u|^2\, dx\,,$$

which can be proved along the lines of [Th, Thm. 4.21]. (The assumptions on V follow easily from Newton's theorem.) Next, integrating the equation against u shows that $\|(-\Delta)^{1/4}u\|^2 + \int V|u|^2\,dx = E\|u\|^2$. Hence,

$$\int_{\mathbb{R}^3} (V + |x|\partial_r V - E)|u|^2\,dx = 0\,.$$

But Newton's theorem gives us

$$V(x) + |x|\partial_r V(x) = -4\pi \int_r^\infty |u(s)|^2 s\,ds \leqslant 0\,,$$

Therefore we have $(V+|x|\partial_r V - E)|u|^2 = 0$ almost everywhere. If $E > 0$, this shows directly that $u \equiv 0$. If $E = 0$ holds, then we conclude $(\int_r^\infty |u(s)|^2 s\,ds)u(r) = 0$ for almost every $r \geqslant 0$, which again implies $u \equiv 0$. This completes the proof of Proposition 1.2. ∎

3. Symmetry

We now establish our first main result of Theorem 1.1. That is, any nonnegative solution $u(x) \geqslant 0$ of problem (1.1) is radially symmetric up to translation. The basic strategy rests on the method of moving planes, which was applied in [GiNiNi2] to obtain a similar statement for the local elliptic equations of the form $-\Delta u + f(u) = 0$. To make the method of moving planes work successfully in our case, we establish a suitable "non-local Hopf lemma" below.

The goal of this section is to establish the following result.

Theorem 3.1 (Symmetry)**.** *Any solution of problem* (1.1) *is radial with respect to some point and strictly decreasing with respect to the distance from that point.*

Since radial symmetry around a point means reflection symmetry with respect to any plane passing through that point, we start by proving a result about reflections. For the sake of concreteness, we consider reflections on the plane $\{x_1 = 0\}$. The following assertion will immediately imply Theorem 3.1.

Proposition 3.2. *Let $u \in H^{1/2}(\mathbb{R}^3)$ be a solution of problem* (1.1) *and assume that the function $f := (u^2 * |x|^{-1})u$ satisfies*

$$\int_{\mathbb{R}^3} y_1 f(y)\,dy = 0\,. \tag{3.1}$$

Then, for each $x' \in \mathbb{R}^2$ fixed, the function $u(\cdot, x')$ is symmetric with respect to the point $x_1 = 0$ and strictly decreasing for $x_1 > 0$.

Before we turn to the proof of Proposition 3.2, we first give the proof of Theorem 3.1.

Proof of Theorem 3.1 assuming Proposition 3.2. Let u be a solution of problem (1.1) and define $f := (u^2 * |x|^{-1})u$. Since $u \geqslant 0$ and $u \not\equiv 0$, we have $\int_{\mathbb{R}^3} f(y)\,dy > 0$. Thus there exists a translation $a \in \mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} y_j f(y-a)\,dy = 0, \qquad \text{for } j = 1,2,3\,. \tag{3.2}$$

(We note that the integrals converge absolutely in view of the estimates from Lemma 2.2.) For any orthogonal matrix $R \in O(3)$, the function $v_R(x) := u(Rx - a)$ is a solution of (1.1) and the normalization (3.2) implies that $f_R := (v_R^2 * |x|^{-1})v_R$

satisfies (3.1). Hence, by Proposition 3.2, the function $v_R(x)$ is symmetric with respect to $x_1 = 0$ and strictly decreasing for $x_1 > 0$. Since the rotation $R \in O(3)$ is arbitrary, this means that u is radial with respect to a and strictly decreasing as a function of $|x - a|$. □

The proof of Proposition 3.2 will be given in Subsection 3.3 after having proved two preliminary results. In this section we use the following notation. For any $\lambda \in \mathbb{R}$ and any point $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^2$, we denote by

$$x^\lambda := (2\lambda - x_1, x') \tag{3.3}$$

its reflection with respect to the hyperplane $\{x_1 = \lambda\}$. Moroever, the reflection of a function u on $\mathbb{R}^3$ with respect to the hyperplane $\{x_1 = \lambda\}$ will be denoted by

$$u_\lambda(x) := u(x^\lambda). \tag{3.4}$$

3.1. **Asymptotics of the solution.** Recall from Lemma 2.2 that any solution u of (1.1) decays like $|x|^{-4}$. To make the method of moving planes work, we need more precise asymptotics of u and its first derivative. To this end, we consider the equations of the following general form:

$$\sqrt{-\Delta}\, u = -u + f, \tag{3.5}$$

where the inhomogeneity $f(x)$ is some given measurable function on $\mathbb{R}^3$. Clearly, this equation coincides with equation in (1.1) if we put $f := (u^2 * |x|^{-1})u$; and according to our a-priori estimates from Lemma 2.2, we then have $0 < f(x) \leqslant C(1+|x|)^{-5}$. In fact, our asymptotics will be valid for more general inhomogeneities $f(x)$. The precise statement is as follows.

Lemma 3.3. *Assume that $u \in H^1(\mathbb{R}^3)$ satisfies (3.5) with $|f(x)| \leqslant C(1 + |x|)^{-\rho}$ for some $\rho > 4$. Then*

(i) $\lim_{|x|\to\infty} |x|^4 u(x) = \pi^{-2} \int f(y)\, dy$.

(ii) $\lim_{x_1\to\infty} \frac{|x|^6}{x_1} \frac{\partial u}{\partial x_1}(x) = -4\pi^{-2} \int f(y)\, dy$.

(iii) *If $\lambda^j \to \lambda$ and $|x^j| \to \infty$ with $x_1^j < \lambda^j$, then*

$$\lim_{j\to\infty} \frac{|x^j|^6}{2(\lambda^j - x_1^j)} \left(u(x^j) - u_{\lambda^j}(x^j)\right) = \frac{4}{\pi^2} \int_{\mathbb{R}^3} f(y)(\lambda - y_1)\, dy\,,$$

where $u_{\lambda^j}(x)$ is defined in (3.4) above.

Proof. We write $u = (\sqrt{-\Delta} + 1)^{-1} f$ and use the explicit formula (2.8) for the resolvent kernel. Calculating the corresponding results for this kernel we easily obtain the assertion of the lemma for f's with compact support. We omit the details. The extension to more general f's uses a density argument in the same spirit as in [GiNiNi2, Lem. 2.1]. □

3.2. **A non-local Hopf lemma.** As a next step, we derive the following non-local Hopf lemma.

Lemma 3.4. *Let $w \in H^1(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ be odd with respect to the plane $\{x_1 = 0\}$ and assume that, for some $\tau \in \mathbb{R}$, we have*

$$\begin{aligned} \sqrt{-\Delta}\, w &\geqslant -\tau w \qquad \text{in } \{x_1 > 0\}\,, \\ w &\geqslant 0 \qquad \text{in } \{x_1 > 0\}\,. \end{aligned} \tag{3.6}$$

Then either $w \equiv 0$, or else $w > 0$ in $\{x_1 > 0\}$ and $\frac{\partial w}{\partial x_1}\big|_{x_1=0} > 0$.

A different extension of Hopf's lemma to the non-local context is proved in [BiLoWa]. Their approach does not allow for positive values of τ which, however, will be crucial for us.

Proof. Since we assume $w \geqslant 0$, it is sufficient to do the proof assuming that $\tau > 0$ holds. Next, we assume that $w \not\equiv 0$ and define $h := (\sqrt{-\Delta} + \tau)w$. We note that h is odd with respect to the plane $\{x_1 = 0\}$ and that $h \geqslant 0$ in $\{x_1 > 0\}$. Moreover, one easily sees that $h \not\equiv 0$; e. g. via the Fourier transform. Next, we write

$$w = (\sqrt{-\Delta} + \tau)^{-1} h = \int_0^\infty e^{-t\tau} \exp(-t\sqrt{-\Delta}) h \, dt \, .$$

This shows that it is enough to prove that $\exp(-t\sqrt{-\Delta})h$ is strictly positive in $\{x_1 > 0\}$ and has a strictly positive x_1-derivative on $\{x_1 = 0\}$.

Using that h is odd with respect to the plane $\{x_1 = 0\}$ and writing $x = (x_1, x')$, $y = (y_1, y')$, we find (recalling formula (2.8) for the integral kernel) that

$$\begin{aligned}
\exp(-t\sqrt{-\Delta})h(x) &= \pi^{-2} \int_{\mathbb{R}^3} \frac{t}{(t^2 + |x-y|^2)^2} h(y) \, dy \\
&= \pi^{-2} \int_{y_1 > 0} \left(\frac{t}{(t^2 + |x-y|^2)^2} \right. \\
&\qquad \left. - \frac{t}{(t^2 + (x_1 + y_1)^2 + |x' - y'|^2)^2} \right) h(y) \, dy \, .
\end{aligned}$$

If $x_1 > 0$, then the integrand is non-negative and $\not\equiv 0$, and hence $\exp(-t\sqrt{-\Delta})h(x) > 0$. Differentiating the above expression under the integral sign (which can be justified by dominated convergence), we find

$$\frac{\partial}{\partial x_1} \exp(-t\sqrt{-\Delta})h(0, x') = \frac{8}{\pi^2} \int_{y_1 > 0} \frac{t y_1}{(t^2 + y_1^2 + |x' - y'|^2)^3} h(y) \, dy \, ,$$

which again is strictly positive. $\square$

3.3. **Proof of Proposition 3.2.** Now we are ready to implement the method of moving planes. Let u be a solution of problem (1.1) and assume that $f := (u^2 * |x|^{-1})u$ satisfies (3.1). Recalling the definition of u_λ before Subsection 3.1, we define the set

$$\Lambda := \{\mu > 0 : \text{ for all } \lambda > \mu \text{ and for all } x \text{ with } x_1 < \lambda \text{ one has } u(x) \geqslant u_\lambda(x)\} \, .$$

We divide the proof of Proposition 3.2 into three steps as follows.

Step 1. Λ is non-empty.

We first note that according to Lemma 3.3 (ii) there is a $\overline{\lambda} > 0$ such that

$$\frac{\partial u}{\partial x_1}(x) < 0 \qquad \text{if } x_1 \geqslant \overline{\lambda} \, . \tag{3.7}$$

We now prove that Λ is non-empty by contradiction. If Λ were empty, there would exist sequences of numbers $(\lambda^j) \to \infty$ and points (x^j) with $x_1^j < \lambda^j$ such that

$$u(x^j) < u_{\lambda^j}(x^j) \, . \tag{3.8}$$

Next, we claim that

$$|x^j| \to \infty \tag{3.9}$$

and, with $\overline{\lambda}$ from (3.7),

$$x_1^j < \overline{\lambda} \, . \tag{3.10}$$

To prove our claim, we note that $(x^j)_1^{\lambda^j} > \lambda^j \to \infty$ together with the decay estimate in Lemma 2.2 implies that $u_{\lambda^j}(x^j) \to 0$. Therefore, by (3.8), we also have $u(x^j) \to 0$. Since u is continuous by Lemma 2.1 and strictly positive by Lemma 2.2, we obtain (3.9). Hence the bound (3.10) follows from (3.7) and (3.8).

Now choose j sufficiently large such that $\lambda^j > \overline{\lambda}$ holds. Then (3.10) implies that $\overline{\lambda} < (x^j)_1^{\overline{\lambda}} < (x^j)_1^{\lambda^j}$. Thus, by (3.7), we conclude

$$u_{\overline{\lambda}}(x^j) > u_{\lambda^j}(x^j)\,. \tag{3.11}$$

On the other hand, (3.9), (3.10) and (3.1) together with Lemma 3.3 (iii) (and $\overline{\lambda}$ instead of λ^j) imply that

$$\frac{|x^j|^6}{2(\overline{\lambda} - x_1^j)} \left(u(x^j) - u_{\overline{\lambda}}(x^j) \right) \to \frac{4\overline{\lambda}}{\pi^2} \int f(y)\, dy > 0\,,$$

contradicting (3.8) and (3.11). Hence the set Λ is non-empty.

Step 2. $\lambda_1 := \inf \Lambda = 0$.

Again, we argue by contradiction and assume that $\lambda_1 > 0$. We note that $u(x) \geqslant u_\lambda(x)$ for all x with $x_1 < \lambda$ and all $\lambda > \lambda_1$. Hence, by continuity (see Lemma 2.1), we also have $u(x) \geqslant u_{\lambda_1}(x)$ if $x_1 < \lambda_1$. Note that the function $w := u - u_{\lambda_1}$ satisfies the equation

$$\sqrt{-\Delta} w + V w = -w + f\,, \qquad V := -\frac{1}{2}(u^2 + u_{\lambda_1}^2) * |x|^{-1}$$

with inhomogeneity

$$f(x) := \frac{1}{2}(u(x) + u_{\lambda_1}(x)) \left((u^2 - u_{\lambda_1}^2) * |x|^{-1} \right)(x).$$

Next, a calculation shows that

$$f(x) = \frac{1}{2}(u(x) + u_{\lambda_1}(x)) \int_{y_1 < \lambda_1} \left(\frac{1}{|x-y|} - \frac{1}{\sqrt{(x_1 + y_1 - 2\lambda_1)^2 + |x' - y'|^2}} \right) \left(u(y)^2 - u_{\lambda_1}(y)^2 \right) dy\,.$$

Since $|x-y| < \sqrt{(x_1 + y_1 - 2\lambda)^2 + |x'-y'|^2}$ if $x_1, y_1 < \lambda_1$, we see that $f \geqslant 0$ in $\{x_1 < \lambda_1\}$ and hence $\sqrt{-\Delta} w \geqslant -w - Vw \geqslant -w$ in that set. Moreover, recall that $w = u - u_{\lambda_1}$ belongs to all $H^s(\mathbb{R}^3)$. Therefore, by the non-local Hopf lemma (Proposition 3.4), we either have $w \equiv 0$, or else

$$w > 0 \text{ in } \{x_1 < \lambda_1\} \qquad \text{and} \qquad \frac{\partial w}{\partial x_1}(x) < 0 \text{ on } \{x_1 = \lambda_1\}\,. \tag{3.12}$$

The first case cannot occur since $w \equiv 0$, $\lambda_1 > 0$ and (3.1) imply $u \equiv 0$, but for $u \equiv 0$ one has $\lambda_1 = 0$.

Hence we will assume that (3.12) holds. By definition of λ_1, there exist sequences of numbers $(\lambda^j) \to \lambda_1-$ and points (x^j) with $x_1^j < \lambda^j$ such that

$$u(x^j) < u_{\lambda^j}(x^j)\,. \tag{3.13}$$

Passing to a subsequence if necessary, we may either assume that $x^j \to x$ or else that $|x^j| \to \infty$.

If $x_j \to x$, then (3.13) implies $u(x) \leqslant u_{\lambda_1}(x)$. Moreover, since $x_1 \leqslant \lambda_1$, the first relation in (3.12) allows us to deduce that $x_1 = \lambda_1$ and $u(x) = u_{\lambda_1}(x)$. Now (3.13) yields $\frac{\partial u}{\partial x_1}(x) \geqslant 0$, contradicting the second relation in (3.12).

If $|x^j| \to \infty$, then we argue as in Step 1 (using Lemma 3.3 (iii) with the sequence (λ^j)) to arrive at a contradiction.

Step 3. Conclusion.

In the previous step we have shown that $u(x) \geqslant u_\lambda(x)$ if $x_1 < \lambda$ for any $\lambda > 0$. Hence by continuity $u(x) \geqslant u(-x_1, x')$ if $x_1 < 0$. Repeating the same argument with $u(x)$ replaced by $u(-x_1, x')$ (and noting that the choice of the origin in (3.1) is not affected by this replacement) yields the reverse inequality $u(-x_1, x') \geqslant u(x)$ if $x_1 < 0$. Hence $u(\cdot, x')$ is symmetric with respect to $x_1 = 0$. Using the nonlocal Hopf lemma (Proposition 3.4) as in Step 2, we find that if $\lambda > 0$ then $u(x) > u_\lambda(x)$ for all $x_1 < \lambda$. This means that $u(\cdot, x')$ is strictly decreasing for $x_1 > 0$. The proof of Proposition 3.2 is complete. ■

4. Real analyticity

In this section, we prove that any real-valued solution of the equation (2.3) is real-analytic, which is a substantial improvement of Lemma 2.1 above. Our proof will derive pointwise exponential decay in Fourier space. A similar argument has been applied in the analyticity proof of solitary waves for some nonlinear water wave equations in $d = 1$ spatial dimension; see [LiBo]. However, apart from higher dimensionality, our case also involves a nonlocal nonlinearity. To deal with this difficulty of a nonlocal nonlinearity, we derive exponential bounds in Fourier space for a coupled system of equations.

Our main result on analyticity is as follows.

Theorem 4.1. *Let $u \in H^{1/2}(\mathbb{R}^3)$ be a real-valued solution of (2.3). Then there exists a constant $\sigma > 0$ and an analytic function $\tilde{u}$ on $\{z \in \mathbb{C}^3 : \ |\operatorname{Im} z_j| < \sigma, 1 \leqslant j \leqslant 3\}$ such that $\tilde{u}(x) = u(x)$ if $x \in \mathbb{R}^3$.*

Note that we do not assume u to be non-negative. Moreover, our proof is independent of the radial symmetry established in the Section 3. We follow the technique developed in [LiBo]. The heart of the argument is contained in the following statement.

Proposition 4.2. *Let $\lambda, \alpha > 0$, $0 \leqslant f \in L_1(\mathbb{R}^3)$ and $0 \leqslant W \in L_1(\mathbb{R}^3, (1+|\xi|)d\xi)$ such that*

$$(|\xi| + \lambda) f \leqslant W * f \,, \qquad |\xi|^2 W \leqslant \alpha f * f \,. \tag{4.1}$$

Then there exist non-negative functions g_n, $n \in \mathbb{N}_0$, and constants $a, b > 0$ such that

$$|\xi|^n f \leqslant g_n * f \,, \qquad \|g_n\|_1 \leqslant a b^n (2n+1)^{n-1} \,. \tag{4.2}$$

In particular, if $f \in L_p(\mathbb{R}^3)$ for some $1 \leqslant p \leqslant \infty$, then

$$\| |\xi|^n f \|_p \leqslant a b^n (2n+1)^{n-1} \|f\|_p \,. \tag{4.3}$$

At several places in this proof we will use the so-called Abel identity,

$$\sum_{l=0}^{n} \binom{n}{l} (l+a)^{l-1} (n-l+b)^{n-l-1} = \frac{a+b}{ab} (n+a+b)^{n-1} \,, \tag{4.4}$$

see [Ri, p. 18].

Proof. We prove (4.2) by induction over n. For $n = 0$, (4.2) follows from (4.1) with $g_0 := \lambda^{-1}W$ and any $a \geqslant \lambda^{-1}\|W\|_1$. Now let $n \geqslant 1$ and assume that (4.2) has already been shown for all smaller values of n. By the triangle inequality one has $|\xi|^{n-1} \leqslant \sum_{l=0}^{n-1}\binom{n-1}{l}|\xi-\eta|^l|\eta|^{n-1}$ and therefore by (4.1) and the induction hypothesis

$$|\xi|^n f \leqslant \sum_{l=0}^{n-1}\binom{n-1}{l}\left(|\eta|^l W\right) * \left(|\eta|^{n-1-l}f\right) \leqslant g_n * f$$

where

$$g_n := \sum_{l=0}^{n-1}\binom{n-1}{l}\left(|\eta|^l W\right) * g_{n-1-l}\,.$$

Hence

$$\|g_n\|_1 \leqslant \sum_{l=0}^{n-1}\binom{n-1}{l}\||\eta|^l W\|_1\|g_{n-1-l}\|_1\,. \tag{4.5}$$

Next, we estimate $\||\eta|^l W\|_1$ for $l \geqslant 2$. For $m \in \mathbb{N}_0$ one has again by (4.1), the triangle inequality and the induction hypothesis for $m < n$

$$|\xi|^{m+2}W \leqslant \alpha\sum_{k=0}^{m}\binom{m}{k}\left(|\eta|^k f\right) * \left(|\eta|^{m-k}f\right) \leqslant \alpha\sum_{k=0}^{m}\binom{m}{k} g_k * f * g_{m-k} * f\,.$$

Hence

$$\begin{aligned}\||\xi|^{m+2}W\|_1 &\leqslant \alpha\|f\|_1^2\sum_{k=0}^{m}\binom{m}{k}\|g_k\|_1\|g_{m-k}\|_1\\ &\leqslant \alpha a^2 b^m\|f\|_1^2\sum_{k=0}^{m}\binom{m}{k}(2k+1)^{k-1}(2(m-k)+1)^{m-k-1}\\ &= 2\alpha a^2 b^m\|f\|_1^2(2m+2)^{m-1}\,,\end{aligned}$$

where we used Abel's identity (4.4) in the last calculation. In order to simplify some arithmetics below, we estimate this by

$$\||\xi|^l W\|_1 \leqslant 2\alpha a^2 b^{l-2}\|f\|_1^2(2l+2)^{l-1} \tag{4.6}$$

for $l \geqslant 2$. If we choose a^2b^{l-2} large enough, then this holds also for $l = 0$ and $l = 1$.

Plugging this into (4.5) and using the induction hypothesis and again Abel's identity, we arrive at

$$\begin{aligned}\|g_n\|_1 &\leqslant 2\alpha a^3 b^{n-3}\|f\|_1^2\sum_{l=0}^{n-1}\binom{n-1}{l}(2l+2)^{l-1}(2(n-1-l)+1)^{n-l-2}\\ &= 3\alpha a^3 b^{n-3}\|f\|_1^2(2n+1)^{n-2}\,.\end{aligned}$$

This proves the assertion provided we have

$$3\alpha a^2\|f\|_1^2 \leqslant b^3(2n+1)\,. \tag{4.7}$$

Let us show that such a choice of parameters a and b is possible. We fix the ratio a/b by $a^2/b^2 = \|W\|_1/(\alpha\|f\|_1^2) =: c^2$ with $c > 0$, so that (4.6) holds for $l = 0$. Now we choose a (keeping the ratio a^2/b^4 fixed) to be

$$a := \max\{\lambda^{-1}\|W\|_1,\ \||\xi|W\|_1/(2\alpha c\|f\|_1^2),\ \alpha c^3\|f\|_1^2\}\,.$$

Hence (4.2) holds for $n = 0$, (4.6) holds for $l = 1$, (4.7) holds for all $n \geqslant 1$ and the proof is complete. □

Proof of Theorem 4.1. Let $u \in H^{1/2}$ be a real-valued solution of (2.3) and $\hat{u}$ its Fourier transform (1.5). Then

$$|\xi|\hat{u} - w * \hat{u} = -\hat{u}$$

with

$$w(\xi) := \frac{1}{(2\pi)^3} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u(y)^2 e^{-i\xi\cdot x}}{|x-y|}\, dx\, dy = \frac{1}{|\xi|^2 2\pi^2} \int_{\mathbb{R}^3} u(y)^2 e^{-i\xi\cdot y}\, dy = \frac{\hat{u} * \hat{u}(\xi)}{2\pi^2|\xi|^2}\,.$$

Here we used that u is real-valued. Hence $f := |\hat{u}|$ satisfies (4.1) with $W := |w|$, $\alpha = (2\pi^2)^{-1}$ and $\lambda = 1$.

We claim that the assumptions of Lemma 4.2 are satisfied. Indeed, by Lemma 2.2, we have $u \in L_1$ and hence $f \in L_\infty$. Also, by Lemma 2.1, we conclude $f \in L_1$. This implies that $\hat{u} * \hat{u} \in L_1 \cap L_\infty$ and hence $W \in L_1(\mathbb{R}^3, (1+|\xi|)d\xi)$. Therefore we can apply Lemma 4.2 and obtain constants a and b such that

$$\sup_\xi |\exp(\tau|\xi|)\hat{u}(\xi)| \leqslant \sum_n \frac{\tau^n}{n!} \||\xi|^n f\|_\infty \leqslant \sum_n \alpha_n \tau^n$$

with $\alpha_n := (n!)^{-1} a b^n (2n+1)^{n-1}\|f\|_\infty$. Since we find

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{b(2n+3)^n}{(n+1)(2n+1)^{n-1}} \to 2be\,,$$

the above supremum is finite for $\tau < \sigma := (2be)^{-1}$. Thus the function

$$\tilde{u}(z) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i(x_1 z_1 + x_2 z_2 + x_3 z_3)} \hat{u}(\xi)\, d\xi$$

is analytic in $\{z \in \mathbb{C}^3 : \ |\operatorname{Im} z_j| < \sigma, 1 \leqslant j \leqslant 3\}$ and it coincides with u on $\mathbb{R}^3$ by Plancherel's theorem. □

Remark 5. As a further corollary of Proposition 4.2, we note that real-analycity also follows for real-valued solutions $u \in H^{1/2}(\mathbb{R}^3)$ satisfying

$$\sqrt{-\Delta + m^2}\, u - \big(u^2 * |x|^{-1}\big)u = -\mu u,$$

where $m > 0$ and $\mu > -m$ are given parameters.

5. Real analyticity II

In this section, we establish (as some additional result) analyticity of kernel elements of the linearized operator associated with Q solving (1.1). Although the arguments will follow closely Section 4, we provide the details of the (tedious) adaptation.

Suppose that $Q \in H^{1/2}(\mathbb{R}^3)$ is a real-valued solution to (2.3). According to Theorem 4.1, Q is real-analytic. We consider the associated linearized operator

$$L_+\xi = \sqrt{-\Delta}\,\xi + \xi - \big(Q^2 * |x|^{-1}\big)\xi - 2Q\big((Q\xi) * |x|^{-1}\big).$$

This defines a self-adjoint operator in $L^2(\mathbb{R}^3)$ with operator domain $H^1(\mathbb{R}^3)$.

We have the following result.

Proposition 5.1. *If $v \in \ker L_+$, then v is real-analytic. More precisely, there exists a constant σ and an analytic function $\tilde{v}$ on $\{z \in \mathbb{C}^3 : |\operatorname{Im} z_j| < \sigma,\ 1 \leqslant j \leqslant 3\}$ such that $\tilde{v}(x) = v(x)$ if $x \in \mathbb{R}^3$.*

The proof of Proposition 5.1 will be given at the end of the section. First, we establish the following auxiliary fact.

Lemma 5.2. *Assume that $v \in \ker L_+$ is radial. Then $v \in L_1$ and hence $\hat{v} \in L_\infty$.*

Proof. We have
$$v = (\sqrt{-\Delta}+1)^{-1}(f_1+f_2)$$
where $f_1 := (Q^2 * |x|^{-1})v$ and $f_2 := 2((Qv)*|x|^{-1})Q$. Let us first consider f_2 and we note that
$$|f_2(x)| \leqslant C(1+|x|)^{-4}|(Qv)*|x|^{-1}|(x) \leqslant C(1+|x|)^{-5},$$
Here, the pointwise bound on Q comes from Lemma 2.2 and the pointwise bound on $(Qv)*|x|^{-1}$ follows from combining Hardy's inequality (to get an L_∞-bound) and Newton's theorem to conclude that $|(Qv)*|x|^{-1}|(x) \leqslant C/|x|$ for $|x|>0$.

Next, we note that $f_1 \in L_{6/5+}$, since $Q^2 * |x|^{-1} \in L_{3+}$ and $v \in L_2$. Since $(\sqrt{-\Delta}+1)^{-1}$ is the convolution by a function in L_1 (indeed, a function bounded by a constant times $\min\{|x|^{-2},|x|^{-4}\}$), we conclude that $v \in L_{6/5+}$. But this implies that $f_1 \in L_1$, and hence v is the convolution of an L_1 kernel with the L_1 function f_1+f_2, and hence v must be in L_1. □

As a next step and similarly as in Section 4, we prove the following statement.

Lemma 5.3. *Let W and V be non-negative functions on $\mathbb{R}^n$ satisfying for some $a,b>0$ and all $n \in \mathbb{N}_0$ and $1 \leqslant p \leqslant \infty$*
$$\||\xi|^n W\|_1 \leqslant ab^n(2n+1)^{n-1}\,, \qquad \||\xi|^n V\|_p \leqslant ab^n(2n+1)^{n-1}\,. \tag{5.1}$$
Let $\lambda>0$, $0 \leqslant f \in L_2(\mathbb{R}^3)\cap L_\infty(\mathbb{R}^3)$ and $g \geqslant 0$ measurable such that
$$(|\xi|+\lambda)f \leqslant W*f+V*g\,, \qquad |\xi|^2 g \leqslant V*f\,. \tag{5.2}$$
Then there exist $\tilde{a},\tilde{b}>0$ such that for all $n \in \mathbb{N}_0$,
$$\||\xi|^n f\|_\infty \leqslant \tilde{a}\tilde{b}^n(2n+1)^{n-1}\,, \qquad \||\xi|^{n+2} g\|_\infty \leqslant \tilde{a}\tilde{b}^{n+2}(2(n+2)+1)^{(n+2)-1}\,. \tag{5.3}$$

Proof. We begin by showing that g and $|\xi|g$ are integrable and that $|\xi|^2 g$ is bounded. To see this, note that since $V \in L_1 \cap L_2$ and $f \in L_2$, $h := |\xi|^2 g \leqslant V*f \in L_2 \cap L_\infty$ and therefore
$$\int_{\mathbb{R}^3} g\,d\xi \leqslant \|h\|_\infty \int_{|\xi|<1} |\xi|^{-2}\,d\xi + \|h\|_2 \left(\int_{|\xi|>1} |\xi|^{-4}\,d\xi\right)^{1/2} < \infty\,.$$
Using this information, as well as $W \in L_1$, $f \in L_2$, $V \in L_2$, we find $|\xi|f \leqslant W*f + V*g \in L_2$. By the triangle inequality, $|\xi|h \leqslant |\xi|(V*f) \leqslant (|\eta|V)*f + V*(|\eta|f) \in L_2 \cap L_\infty$, and therefore
$$\int_{\mathbb{R}^3} |\xi|g\,d\xi \leqslant \|h\|_\infty \int_{|\xi|<1} |\xi|^{-1}\,d\xi + \||\xi|h\|_2 \left(\int_{|\xi|>1} |\xi|^{-4}\,d\xi\right)^{1/2} < \infty\,.$$

We define
$$\tilde{a} := \max\{\|f\|_\infty, \|g\|_1\}\,.$$
Since (5.1) remains true if b is increased, we may assume that
$$b \geqslant \max\{\left(\||\xi|^2 g\|_\infty/(5\tilde{a})\right)^{1/2}, \||\xi|g\|_1/\tilde{a}, 2\tilde{a}, (2\tilde{a})^{1/2}/7\}$$

Note that these choices imply that

$$\||\xi|^l g\|_1 \leqslant \tilde{a} b^l (2l+1)^{l-1} \qquad \text{for } l = 0, 1\,. \tag{5.4}$$

Having modified b in this way, we shall prove (5.3) with $\tilde{b} = b$ (and $\tilde{a}$ as defined above). We proceed by induction with respect to $n \in \mathbb{N}_0$. For $n = 0$ the assertion is an immediate consequence of our choices for $\tilde{a}$ and b. Now let $n \geqslant 1$ and assume that (5.3) has already been shown for all smaller values of n. By (5.2) and the triangle inequality

$$|\xi|^n f \leqslant \sum_{l=0}^{n-1} \binom{n-1}{l} \left(\left(|\eta|^l W\right) * \left(|\eta|^{n-1-l} f\right) + \left(|\eta|^l V\right) * \left(|\eta|^{n-1-l} g\right) \right)$$

and therefore

$$\begin{aligned}\||\xi|^n f\|_\infty \leqslant \sum_{l=0}^{n-1} \binom{n-1}{l} \||\xi|^l W\|_1 \||\xi|^{n-1-l} f\|_\infty + \sum_{l=0}^{n-3} \binom{n-1}{l} \||\xi|^l V\|_1 \||\xi|^{n-1-l} g\|_\infty \\ + \sum_{l=n-2}^{n-1} \binom{n-1}{l} \||\xi|^l V\|_\infty \||\xi|^{n-1-l} g\|_1\,.\end{aligned}$$

(The middle sum should be discarded if $n \leqslant 2$.) Hence by the induction hypothesis, by (5.1) and (5.4)

$$\||\xi|^n f\|_\infty \leqslant 2a\tilde{a}b^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (2l+1)^{l-1} (2(n-l-1)+1)^{n-l-2} = 4a\tilde{a}b^{n-1}(2n)^{n-2}\,.$$

In the last calculation we used Abel's identity (4.4). This proves the first assertion in (5.3), provided we have

$$4a\tilde{a}b^{n-1}(2n)^{n-2} \leqslant ab^n (2n+1)^{n-1}$$

for all $n \geqslant 1$. It is easy to see that this holds if $2\tilde{a} \leqslant b$, which holds by the choice of b.

To prove the second assertion in (5.3) we proceed similarly as before, using the triangle inequality to get

$$|\xi|^{n+2} g \leqslant \sum_{l=0}^{n} \binom{n}{l} \left(|\eta|^l V\right) * \left(|\eta|^{n-l} f\right)$$

and hence

$$\||\xi|^{n+2} g\|_\infty \leqslant \sum_{l=0}^{n} \binom{n}{l} \||\eta|^l V\|_1 \||\eta|^{n-l} f\|_\infty\,.$$

Relation (5.1) together with the estimates for $|\xi|^{n-l} f$ (which we have already proved) and Abel's identity (4.4) imply that

$$\||\xi|^{n+2} g\|_\infty \leqslant a\tilde{a}b^n \sum_{l=0}^{n} \binom{n}{l} (2l+1)^{l-1} (2(n-l)+1)^{n-l-1} = 2a\tilde{a}b^n (2n+2)^{n-1}\,.$$

This proves the second assertion in (5.3), provided we have

$$2a\tilde{a}b^n (2n+2)^{n-1} \leqslant ab^{n+2} (2(n+2)+1)^{(n+2)-1}$$

for all $n \geqslant 1$. It is easy to see that this holds if $\tilde{a} \leqslant 49b^2/2$, which holds by the choice of b. This completes the proof of Lemma 5.3. $\square$

Proof of Proposition 5.1. For the Fourier transform $\hat{v}$, the equation $L_+ v = 0$ leads to

$$|\xi|\hat{v} - w * \hat{v} - \tilde{w} * \hat{Q} = -\hat{v}$$

with w as in the proof of Proposition 4.1 and

$$\tilde{w}(\xi) := \frac{\hat{Q} * \hat{v}(\xi)}{\pi^2 |\xi|^2} .$$

Hence $f := |\hat{v}|$ and $g := |\tilde{w}|$ satisfy (5.2) with $W := |w|$ and $V := \pi^{-1}|\hat{Q}|$. (Indeed, we know that $\hat{Q} > 0$, but we do not need this fact.) Now the assumptions (5.1) can be deduced from (4.3) and (4.6) after modifying a and b. (Strictly speaking, we use the equation before (4.6) which yields the term $(2l+2)$ in (4.6) replaced by $(2l+1)$.) Moreover, $f \in L_\infty$ by Lemma 5.2. Now the statement of Proposition 5.1 follows as in the proof of Theorem 4.1. □

References

[BiLoWa] M. Birkner, J. A. López-Mimbela, A. Wakolbinger, *Comparison results and steady states for the Fujita equation with fractional Laplacian.* Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 1, 83–97.

[BrLiLu] H. J. Brascamp, E. H. Lieb, J. M. Luttinger, *A general rearrangement inequality for multiple integrals,* J. Functional Analysis **17** (1974), 227–237.

[CaMaSi] R. Carmona, W. C. Masters, B. Simon, *Relativistic Schrödinger operators: asymptotic behavior of the eigenfunctions*, J. Funct. Anal. **91** (1990), no. 1, 117–142.

[ChLiOu] W. Chen, C. Li, B. Ou, *Classification for solutions of an integral equation.* Comm. Pure Appl. Math. **59** (2006), no. 3, 330-343.

[ElSc] A. Elgart, B. Schlein, *Mean field dynamics of boson stars*, Comm. Pure Appl. Math. **60** (2007), no. 4, 500–545.

[Ev] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 1998.

[FrSe] R. L. Frank, R. Seiringer, *Non-linear ground state representations and sharp Hardy inequalities.* J. Funct. Anal. **255** (2008), 3407–3430.

[FraLe] R. L. Frank, E. Lenzmann, *Uniqueness and nondegeneracy of ground states for* $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ *in* $\mathbb{R}$. Preprint available at `arXiv:1009.4042`.

[FrLe] J. Fröhlich, E. Lenzmann, *Blowup for nonlinear wave equations describing boson stars*, Comm. Pure Appl. Math. **60** (2007), no. 11, 1691–1705.

[GiNiNi] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[GiNiNi2] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* R^n. In: Mathematical analysis and applications, Part A, pp. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.

[He] I. W. Herbst, *Spectral theory of the operator* $(p^2+m^2)^{1/2} - Ze^2/r$. Comm. Math. Phys. **53** (1977), no. 3, 285–294.

[Le2] E. Lenzmann, *Well-posedness for semi-relativistic Hartree equations of critical type*, Math. Phys. Anal. Geom. **10** (2007), no. 1, 43–64.

[Le3] E. Lenzmann, *Nonlinear dispersive equations describing Boson stars*, ETH Dissertation No. 16572 (2006).

[LiBo] Y. A. Li, J. L. Bona, *Analyticity of solitary-wave solutions of model equations for long waves.* SIAM J. Math. Anal. **27** (1996), no. 3, 725–737.

[Li] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation.* Studies in Appl. Math. **57** (1977), no. 2, 93–105.

[LiLo] E. H. Lieb, M. Loss, *Analysis. Second edition.* Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, 2001.

[LiOx] E. H. Lieb, S. Oxford, *An improved lower bound on the indirect Coulomb energy.* Int. J. Quant. Chem. **19** (1981), 427–439.

[LiYa] E. H. Lieb, H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics.* Comm. Math. Phys. **112** (1987), no. 1, 147–174.

[MaZh] L. Ma, L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*. To appear in Arch. Rat. Mech. (2009).

[Ri] J. Riordan, *Combinatorial identities*, John Wiley & Sons, New York, 1968.

[Th] B. Thaller, *The Dirac equation*. Texts and Monographs in Physics. Springer, Berlin, 1992.

Rupert L. Frank, Department of Mathematics, Princeton University, Washington Road, Princeton, NJ 08544, USA

E-mail address: `rlfrank@math.princeton.edu`

Enno Lenzmann, Department of Mathematical Sciences, University of Copenhagen, Universitetspark 5, 2100 Copenhagen Ø, Denmark

E-mail address: `lenzmann@math.ku.dk`